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The Four Faces of Mathematics

Keith Devlin

We speak of mathematics as if it were a single entity, but in fact it has four faces that it presents to the world:

- Computation, formal reasoning, and problem solving.
- Mathematics as a way of knowing.
- Mathematics as a creative medium.
- Applications.

Most school (and university) education concentrates on the first face and makes some reference to the fourth face, but pays little or no attention to the remaining two. I believe that, as we enter the twenty-first century, mathematics education should display all four faces. Here's why.

Start with the question: What is the purpose of a school education? In a world where the pace of change is such that no one can predict more than a few years in advance what the most important knowledge and skills are going to be, a major goal of education is to develop in young people the ability to acquire specialized knowledge and skills as and when they need them throughout their working lives.

But I also think that (at least in a relatively affluent modern society) preparation for work is just one part of the real purpose of a school education. Arguably the main purpose is to pass on the main elements of our culture and to prepare young people to lead full and active lives, to play their full roles as citizens, and to be productive in their own terms.

What does this imply for mathematics education? Well, given the importance of mathematics in modern society, we cannot afford to ignore the "pipeline" issue. Whatever else we do with mathematics education, we must ensure that those who find they have an interest in, and aptitude for, the subject are able to study it in depth.

For the rest -- and that means the vast majority of students in school mathematics classes -- a major goal should be to create an awareness of the nature of mathematics and the role it plays in contemporary society. This means that school mathematics should be taught much more like history or geography or English literature -- not as a utilitarian

toolbox but as a part of human culture. An educated citizen should be able to answer the following two questions about mathematics:

- What is mathematics?

- Where and how is mathematics used?

At present, few people can answer either question completely. It is in order to provide our students with a broad view of mathematics as a part of human culture that we need to address all four faces of mathematics. How do we go about this?

The familiar face
The first face of mathematics (computation, formal reasoning, and problem solving) is the one we generally concentrate on when we think of mathematics education. Although the pocket calculator and the computer have made redundant the need to spend large amounts of time developing skill in mental arithmetic, basic arithmetic and a good sense of number are surely invaluable skills for everyone to have in today's world.

Mathematics as a way of knowing
Mathematics is a particular way of knowing, a way of understanding different aspects of the world we live in. Today, scarcely any aspect of our lives is not affected, often in a fundamental and far-reaching way, by the products of mathematics. When you think of the technological and communications infrastructure in which we now live our lives, you realize that we are in fact living in a "mathematical universe." And yet most people think that mathematics is merely a collection of rules for manipulating numbers. But this is just one very small part of mathematics.

The simplest, most accurate description of mathematics I know is this: Mathematics is the science of patterns. The mathematician looks at a certain aspect of the world and strips away the complexity, leaving an underlying skeleton. I believe it is important to make our students aware of some of the many different parts to the subject, and what it is about them that makes them all mathematics. We do so by describing some of mathematics' many applications. In so doing, we show our students that mathematics works by *making the invisible visible*. We show them that, by providing us with a means to "see" (and hence to understand) things that would otherwise be invisible, mathematics demonstrates itself to be one of the most amazing constructions of the human mind. Here are just a few examples of the kind of thing I have in mind.

Without mathematics, there is no way you can understand what keeps a jumbo jet in the air. It takes mathematics to "see" what keeps an airplane aloft. In this case, what lets you "see" the invisible is an equation discovered by the mathematician Daniel Bernoulli early in the eighteenth century.

Similarly, Newton's mathematics enables us to "see" the invisible force (gravity) that keeps the earth rotating around the sun and causes an apple to fall from the tree onto the ground.

Probability theory and mathematical statistics help us to predict the outcomes of elections, often with remarkable accuracy.

We use calculus to predict tomorrow's weather.

Insurance companies use statistics and probability theory to predict the likelihood of an accident during the coming year, and set their premiums accordingly.

Let me stress that I am not advocating that we teach our students how to perform the mathematics involved in such applications. Rather, alongside the parts of mathematics that we do require our students to carry out, we should also describe some of the many other (perhaps more advanced) parts of the subject and the different ways they are applied. Just as you don't have to know how to build or repair a car in order to take a tour in the country, so too you don't need to know how to do mathematics in order to see how it is used.

Mathematics as a creative medium

Few people are aware of the breadth of modern mathematics. Even fewer people realize that mathematics can also be used as a creative medium, in much the same way that a sculptor uses stone, a painter uses paint and canvas, or a novelist uses language.

Arguably the first major use of mathematics in a creative fashion occurred in the Renaissance, when artists discovered how to show depth in a two-dimensional painting. Artists refer to the trick as the rules of perspective; mathematicians call it projective geometry. Whatever your preferred terminology, the underlying idea is to discover and utilize a "geometry" -- the geometry of vision.

In a similar vein, present day artists have learned to use a geometry of the way light travels (so-called "ray tracing") to produce realistic looking computer graphics images for the movie industry, images that are complete with surface texture, highlighting, and shadow.

More generally, much of the digital special effects work in today's movie industry amounts to the use of mathematics as a creative medium.

The artist Tony Robbin (see Devlin 1998) has spent a large part of his career trying to depict four-dimensional space on a two-dimensional canvas -- a sort of "super-perspective," if you will. According to Robbin, one of the main functions of art is to reflect on, comment on, and thereby help us to understand, various aspects of life. He sees his own work exploring higher dimensional spaces as a way to visualize and understand the multidimensional complexities of life in multiracial and multicultural societies.

Applications

Mathematics education must include applications. In order to live a full life, everyone needs to have an awareness of what goes into making that life possible. When we travel

by car, train, or airplane, we enter a world that depends on mathematics. When we pick up a telephone, or watch television, or go to a movie, or attend a major sporting event, we are enjoying the products of mathematics. When we listen to music on a CD, log on to the Internet, or cook our meal in a microwave oven, we are using the products of mathematics. When we go into hospital, or take out insurance, or check the weather forecast, we are dependent on mathematics. As educators, we owe it to our students to make them aware of the scope, the depth, and the profound impact of applications of mathematics in today's world.

Of course, much of the mathematics that lies behind our everyday world is advanced and highly specialized. Consequently, most classroom discussions of the applications of mathematics will have to be just that: discussions. But not all applications of mathematics are inaccessible. With improving computer technology, it is getting easier for us to have our students actually carry out some (well chosen) applications of mathematics. Thus, we are not restricted solely to "talking about" applications; we can get our students to carry out some applications.

Bibliography

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